

$$\begin{aligned}
 p_{3,x} &= -m_1 v_1 - m_2 v_2 \cos \theta \\
 &= -(14.5 \text{ kg})(105 \frac{\text{m}}{\text{s}}) - (4.5 \text{ kg})(77 \frac{\text{m}}{\text{s}}) \cos(19^\circ) \\
 &= -1850 \frac{\text{kg} \cdot \text{m}}{\text{s}}.
 \end{aligned}$$

For the y-direction, we have

$$\begin{aligned}
 p_{3y} &= 0 - m_2 v_2 \sin \theta \\
 &= -(4.5 \text{ kg})(77 \frac{\text{m}}{\text{s}}) \sin(19^\circ) \\
 &= -113 \frac{\text{kg} \cdot \text{m}}{\text{s}}.
 \end{aligned}$$

This gives the magnitude of  $p_3$ :

$$\begin{aligned}
 p_3 &= \sqrt{p_{3,x}^2 + p_{3,y}^2} \\
 &= \sqrt{\left(-1850 \frac{\text{kg} \cdot \text{m}}{\text{s}}\right)^2 + \left(-113 \frac{\text{kg} \cdot \text{m}}{\text{s}}\right)^2} \\
 &= 1854 \frac{\text{kg} \cdot \text{m}}{\text{s}}.
 \end{aligned}$$

The velocity of the third piece is therefore

$$v_3 = \frac{p_3}{m_3} = \frac{1854 \frac{\text{kg} \cdot \text{m}}{\text{s}}}{6.7 \text{ kg}} = 277 \frac{\text{m}}{\text{s}}.$$

The direction of its velocity vector is the same as the direction of its momentum vector:

$$\phi = \tan^{-1} \left( \frac{p_{3,y}}{p_{3,x}} \right) = \tan^{-1} \left( \frac{113 \frac{\text{kg} \cdot \text{m}}{\text{s}}}{1850 \frac{\text{kg} \cdot \text{m}}{\text{s}}} \right) = 3.5^\circ.$$

Because  $\phi$  is below the  $-x$ -axis, the actual angle is  $183.5^\circ$  from the  $+x$ -direction.

### Significance

The enormous velocities here are typical; an exploding tank of any compressed gas can easily punch through the wall of a house and cause significant injury, or death. Fortunately, such explosions are extremely rare, on a percentage basis.



**9.10 Check Your Understanding** Notice that the mass of the air in the tank was neglected in the analysis and solution. How would the solution method changed if the air was included? How large a difference do you think it would make in the final answer?

## 9.6 | Center of Mass

### Learning Objectives

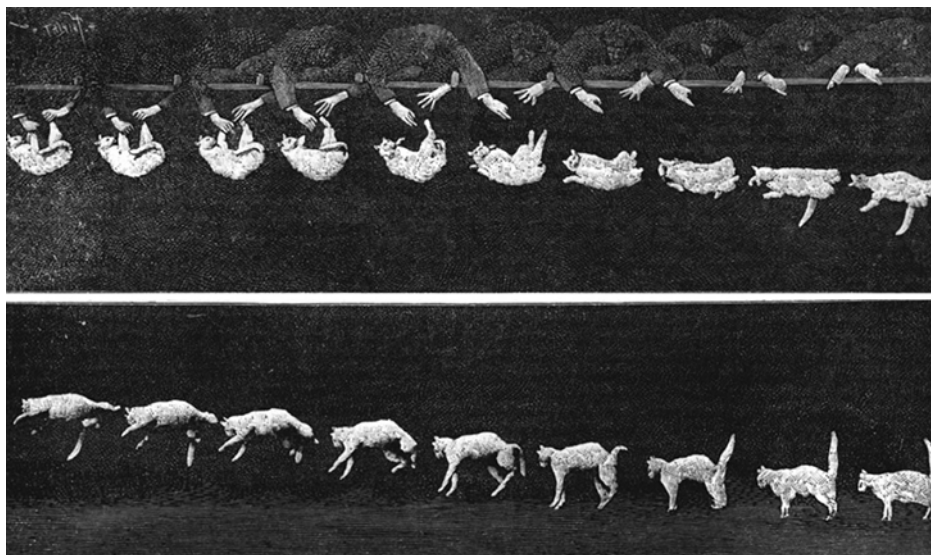
By the end of this section, you will be able to:

- Explain the meaning and usefulness of the concept of center of mass
- Calculate the center of mass of a given system
- Apply the center of mass concept in two and three dimensions
- Calculate the velocity and acceleration of the center of mass

We have been avoiding an important issue up to now: When we say that an object moves (more correctly, accelerates) in a way that obeys Newton's second law, we have been ignoring the fact that all objects are actually made of many constituent

particles. A car has an engine, steering wheel, seats, passengers; a football is leather and rubber surrounding air; a brick is made of atoms. There are many different types of particles, and they are generally not distributed uniformly in the object. How do we include these facts into our calculations?

Then too, an extended object might change shape as it moves, such as a water balloon or a cat falling (**Figure 9.26**). This implies that the constituent particles are applying internal forces on each other, in addition to the external force that is acting on the object as a whole. We want to be able to handle this, as well.



**Figure 9.26** As the cat falls, its body performs complicated motions so it can land on its feet, but one point in the system moves with the simple uniform acceleration of gravity.

The problem before us, then, is to determine what part of an extended object is obeying Newton’s second law when an external force is applied and to determine how the motion of the object as a whole is affected by both the internal and external forces.

Be warned: To treat this new situation correctly, we must be rigorous and completely general. We won’t make any assumptions about the nature of the object, or of its constituent particles, or either the internal or external forces. Thus, the arguments will be complex.

## Internal and External Forces

Suppose we have an extended object of mass  $M$ , made of  $N$  interacting particles. Let’s label their masses as  $m_j$ , where  $j = 1, 2, 3, \dots, N$ . Note that

$$M = \sum_{j=1}^N m_j. \quad (9.19)$$

If we apply some net **external force**  $\vec{F}_{\text{ext}}$  on the object, every particle experiences some “share” or some fraction of that external force. Let:

$$\vec{f}_j^{\text{ext}} = \text{the fraction of the external force that the } j\text{th particle experiences.}$$

Notice that these fractions of the total force are not necessarily equal; indeed, they virtually never are. (They *can* be, but they usually aren’t.) In general, therefore,

$$\vec{f}_1^{\text{ext}} \neq \vec{f}_2^{\text{ext}} \neq \dots \neq \vec{f}_N^{\text{ext}}.$$

Next, we assume that each of the particles making up our object can interact (apply forces on) every other particle of the object. We won’t try to guess what kind of forces they are; but since these forces are the result of particles of the object acting on other particles of the same object, we refer to them as **internal forces**  $\vec{f}_j^{\text{int}}$ ; thus:

$\vec{\mathbf{f}}_j^{\text{int}}$  = the net internal force that the  $j$ th particle experiences from all the other particles that make up the object.

Now, the *net* force, internal plus external, on the  $j$ th particle is the vector sum of these:

$$\vec{\mathbf{f}}_j = \vec{\mathbf{f}}_j^{\text{int}} + \vec{\mathbf{f}}_j^{\text{ext}}. \quad (9.20)$$

where again, this is for all  $N$  particles;  $j = 1, 2, 3, \dots, N$ .

As a result of this fractional force, the momentum of each particle gets changed:

$$\begin{aligned} \vec{\mathbf{f}}_j &= \frac{d\vec{\mathbf{p}}_j}{dt} \\ \vec{\mathbf{f}}_j^{\text{int}} + \vec{\mathbf{f}}_j^{\text{ext}} &= \frac{d\vec{\mathbf{p}}_j}{dt}. \end{aligned} \quad (9.21)$$

The net force  $\vec{\mathbf{F}}$  on the *object* is the vector sum of these forces:

$$\begin{aligned} \vec{\mathbf{F}}_{\text{net}} &= \sum_{j=1}^N (\vec{\mathbf{f}}_j^{\text{int}} + \vec{\mathbf{f}}_j^{\text{ext}}) \\ &= \sum_{j=1}^N \vec{\mathbf{f}}_j^{\text{int}} + \sum_{j=1}^N \vec{\mathbf{f}}_j^{\text{ext}}. \end{aligned} \quad (9.22)$$

This net force changes the momentum of the object as a whole, and the net change of momentum of the object must be the vector sum of all the individual changes of momentum of all of the particles:

$$\vec{\mathbf{F}}_{\text{net}} = \sum_{j=1}^N \frac{d\vec{\mathbf{p}}_j}{dt}. \quad (9.23)$$

Combining **Equation 9.22** and **Equation 9.23** gives

$$\sum_{j=1}^N \vec{\mathbf{f}}_j^{\text{int}} + \sum_{j=1}^N \vec{\mathbf{f}}_j^{\text{ext}} = \sum_{j=1}^N \frac{d\vec{\mathbf{p}}_j}{dt}. \quad (9.24)$$

Let's now think about these summations. First consider the internal forces term; remember that each  $\vec{\mathbf{f}}_j^{\text{int}}$  is the force on the  $j$ th particle from the other particles in the object. But by Newton's third law, for every one of these forces, there must be another force that has the same magnitude, but the opposite sign (points in the opposite direction). These forces do not cancel; however, that's not what we're doing in the summation. Rather, we're simply *mathematically adding up* all the internal force vectors. That is, in general, the internal forces for any individual part of the object won't cancel, but when all the internal forces are added up, the internal forces must cancel in pairs. It follows, therefore, that the sum of all the internal forces must be zero:

$$\sum_{j=1}^N \vec{\mathbf{f}}_j^{\text{int}} = 0.$$

(This argument is subtle, but crucial; take plenty of time to completely understand it.)

For the external forces, this summation is simply the total external force that was applied to the whole object:

$$\sum_{j=1}^N \vec{\mathbf{f}}_j^{\text{ext}} = \vec{\mathbf{F}}_{\text{ext}}.$$

As a result,

$$\vec{\mathbf{F}}_{\text{ext}} = \sum_{j=1}^N \frac{d\vec{\mathbf{p}}_j}{dt}. \quad (9.25)$$

This is an important result. **Equation 9.25** tells us that the total change of momentum of the entire object (all  $N$  particles) is due only to the external forces; the internal forces do not change the momentum of the object as a whole. This is why you can't lift yourself in the air by standing in a basket and pulling up on the handles: For the system of you + basket, your upward pulling force is an internal force.

## Force and Momentum

Remember that our actual goal is to determine the equation of motion for the entire object (the entire system of particles). To that end, let's define:

$\vec{\mathbf{p}}_{\text{CM}}$  = the total momentum of the system of  $N$  particles (the reason for the subscript will become clear shortly)

Then we have

$$\vec{\mathbf{p}}_{\text{CM}} \equiv \sum_{j=1}^N \vec{\mathbf{p}}_j$$

and therefore **Equation 9.25** can be written simply as

$$\vec{\mathbf{F}} = \frac{d \vec{\mathbf{p}}_{\text{CM}}}{dt}. \quad (9.26)$$

Since this change of momentum is caused by only the net external force, we have dropped the “ext” subscript.

This is Newton's second law, but now for the entire extended object. If this feels a bit anticlimactic, remember what is hiding inside it:  $\vec{\mathbf{p}}_{\text{CM}}$  is the vector sum of the momentum of (in principle) hundreds of thousands of billions of billions of particles ( $6.02 \times 10^{23}$ ), all caused by one simple net external force—a force that you can calculate.

## Center of Mass

Our next task is to determine what part of the extended object, if any, is obeying **Equation 9.26**.

It's tempting to take the next step; does the following equation mean anything?

$$\vec{\mathbf{F}} = M \vec{\mathbf{a}} \quad (9.27)$$

If it *does* mean something (acceleration of what, exactly?), then we could write

$$M \vec{\mathbf{a}} = \frac{d \vec{\mathbf{p}}_{\text{CM}}}{dt}$$

and thus

$$M \vec{\mathbf{a}} = \sum_{j=1}^N \frac{d \vec{\mathbf{p}}_j}{dt} = \frac{d}{dt} \sum_{j=1}^N \vec{\mathbf{p}}_j$$

which follows because the derivative of a sum is equal to the sum of the derivatives.

Now,  $\vec{\mathbf{p}}_j$  is the momentum of the  $j$ th particle. Defining the positions of the constituent particles (relative to some coordinate system) as  $\vec{\mathbf{r}}_j = (x_j, y_j, z_j)$ , we thus have

$$\vec{\mathbf{p}}_j = m_j \vec{\mathbf{v}}_j = m_j \frac{d \vec{\mathbf{r}}_j}{dt}.$$

Substituting back, we obtain

$$\begin{aligned}
 M \vec{a} &= \frac{d}{dt} \sum_{j=1}^N m_j \frac{d \vec{r}_j}{dt} \\
 &= \frac{d^2}{dt^2} \sum_{j=1}^N m_j \vec{r}_j.
 \end{aligned}$$

Dividing both sides by  $M$  (the total mass of the extended object) gives us

$$\vec{a} = \frac{d^2}{dt^2} \left( \frac{1}{M} \sum_{j=1}^N m_j \vec{r}_j \right). \quad (9.28)$$

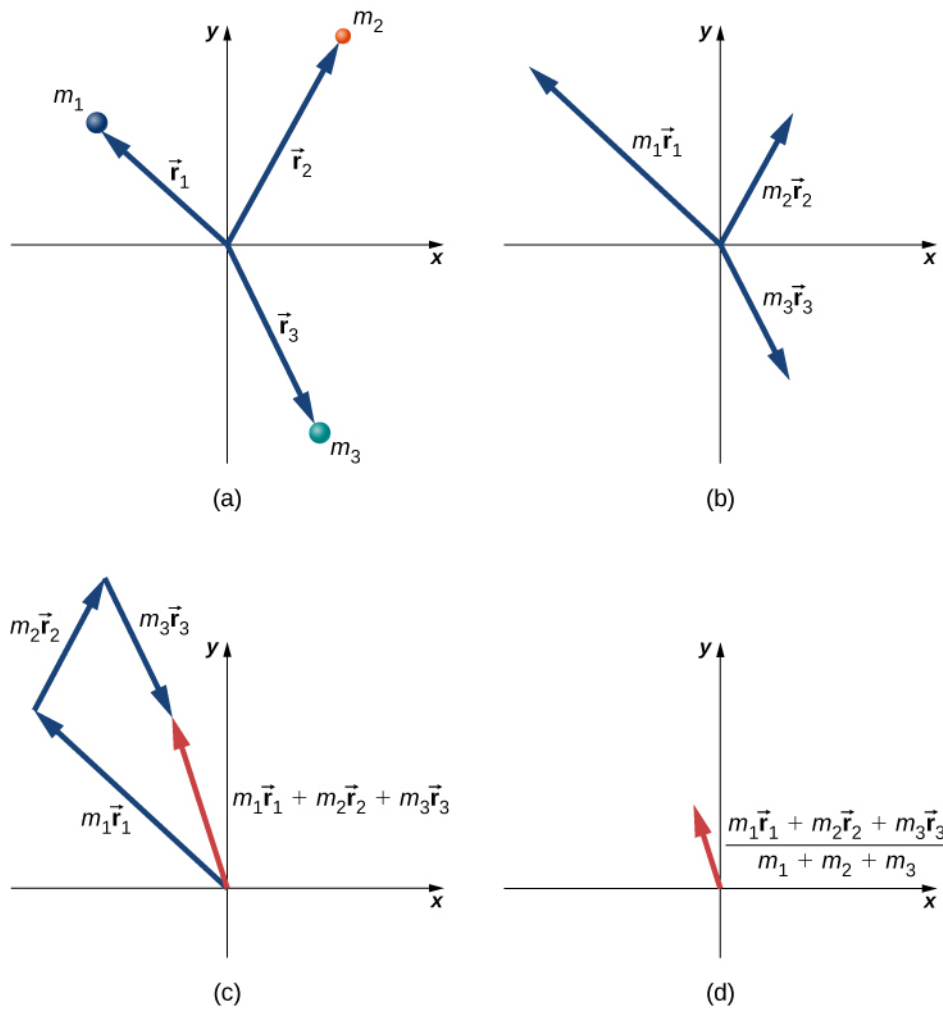
Thus, the point in the object that traces out the trajectory dictated by the applied force in **Equation 9.27** is inside the parentheses in **Equation 9.28**.

Looking at this calculation, notice that (inside the parentheses) we are calculating the product of each particle's mass with its position, adding all  $N$  of these up, and dividing this sum by the total mass of particles we summed. This is reminiscent of an average; inspired by this, we'll (loosely) interpret it to be the weighted average position of the mass of the extended object. It's actually called the **center of mass** of the object. Notice that the position of the center of mass has units of meters; that suggests a definition:

$$\vec{r}_{\text{CM}} \equiv \frac{1}{M} \sum_{j=1}^N m_j \vec{r}_j. \quad (9.29)$$

So, the point that obeys **Equation 9.26** (and therefore **Equation 9.27** as well) is the center of mass of the object, which is located at the position vector  $\vec{r}_{\text{CM}}$ .

It may surprise you to learn that there does not have to be any actual mass at the center of mass of an object. For example, a hollow steel sphere with a vacuum inside it is spherically symmetrical (meaning its mass is uniformly distributed about the center of the sphere); all of the sphere's mass is out on its surface, with no mass inside. But it can be shown that the center of mass of the sphere is at its geometric center, which seems reasonable. Thus, there is no mass at the position of the center of mass of the sphere. (Another example is a doughnut.) The procedure to find the center of mass is illustrated in **Figure 9.27**.



**Figure 9.27** Finding the center of mass of a system of three different particles. (a) Position vectors are created for each object. (b) The position vectors are multiplied by the mass of the corresponding object. (c) The scaled vectors from part (b) are added together. (d) The final vector is divided by the total mass. This vector points to the center of mass of the system. Note that no mass is actually present at the center of mass of this system.

Since  $\vec{r}_j = x_j \hat{i} + y_j \hat{j} + z_j \hat{k}$ , it follows that:

$$r_{CM,x} = \frac{1}{M} \sum_{j=1}^N m_j x_j \quad (9.30)$$

$$r_{CM,y} = \frac{1}{M} \sum_{j=1}^N m_j y_j \quad (9.31)$$

$$r_{CM,z} = \frac{1}{M} \sum_{j=1}^N m_j z_j \quad (9.32)$$

and thus

$$\vec{r}_{CM} = r_{CM,x} \hat{i} + r_{CM,y} \hat{j} + r_{CM,z} \hat{k}$$

$$r_{CM} = |\vec{r}_{CM}| = \left( r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2 \right)^{1/2}.$$

Therefore, you can calculate the components of the center of mass vector individually.

Finally, to complete the kinematics, the instantaneous velocity of the center of mass is calculated exactly as you might suspect:

$$\vec{v}_{\text{CM}} = \frac{d}{dt} \left( \frac{1}{M} \sum_{j=1}^N m_j \vec{r}_j \right) = \frac{1}{M} \sum_{j=1}^N m_j \vec{v}_j \quad (9.33)$$

and this, like the position, has x-, y-, and z-components.

To calculate the center of mass in actual situations, we recommend the following procedure:

### Problem-Solving Strategy: Calculating the Center of Mass

The center of mass of an object is a position vector. Thus, to calculate it, do these steps:

1. Define your coordinate system. Typically, the origin is placed at the location of one of the particles. This is not required, however.
2. Determine the x, y, z-coordinates of each particle that makes up the object.
3. Determine the mass of each particle, and sum them to obtain the total mass of the object. Note that the mass of the object at the origin *must* be included in the total mass.
4. Calculate the x-, y-, and z-components of the center of mass vector, using **Equation 9.30**, **Equation 9.31**, and **Equation 9.32**.
5. If required, use the Pythagorean theorem to determine its magnitude.

Here are two examples that will give you a feel for what the center of mass is.

## Example 9.16

### Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

#### Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use **Equation 9.29** with just  $N = 2$  objects. We use a subscript “e” to refer to Earth, and subscript “m” to refer to the moon.

#### Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, **Equation 9.29** becomes

$$R = \frac{m_e r_e + m_m r_m}{m_e + m_m}.$$

From **Appendix D**,

$$m_e = 5.97 \times 10^{24} \text{ kg}$$

$$m_m = 7.36 \times 10^{22} \text{ kg}$$

$$r_m = 3.82 \times 10^8 \text{ m}.$$

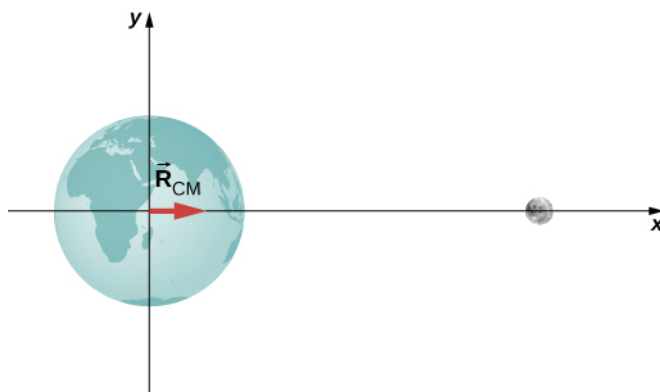
We defined the center of Earth as the origin, so  $r_e = 0 \text{ m}$ . Inserting these into the equation for  $R$  gives

$$R = \frac{(5.97 \times 10^{24} \text{ kg})(0 \text{ m}) + (7.36 \times 10^{22} \text{ kg})(3.82 \times 10^8 \text{ m})}{5.97 \times 10^{24} \text{ kg} + 7.36 \times 10^{22} \text{ kg}}$$

$$= 4.64 \times 10^6 \text{ m.}$$

### Significance

The radius of Earth is  $6.37 \times 10^6 \text{ m}$ , so the center of mass of the Earth-moon system is  $(6.37 - 4.64) \times 10^6 \text{ m} = 1.73 \times 10^6 \text{ m} = 1730 \text{ km}$  (roughly 1080 miles) *below* the surface of Earth. The location of the center of mass is shown (not to scale).



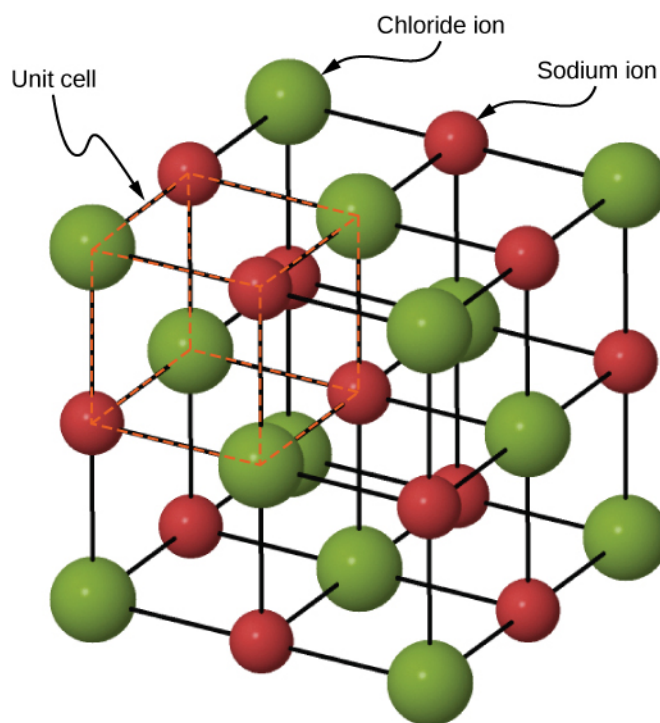
**9.11 Check Your Understanding** Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

## Example 9.17

### Center of Mass of a Salt Crystal

**Figure 9.28** shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the *unit cell*) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is  $2.36 \times 10^{-10} \text{ m}$ . Find the location of the center of mass of the unit cell. Specify it either by its coordinates  $(r_{CM,x}, r_{CM,y}, r_{CM,z})$ , or by  $r_{CM}$  and two angles.





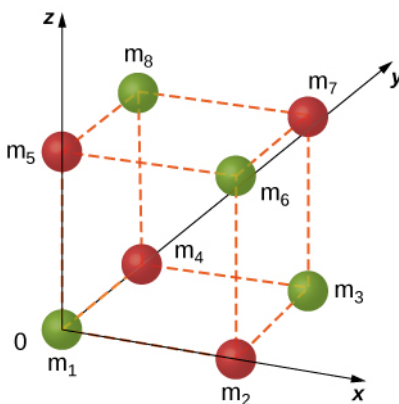
**Figure 9.28** A drawing of a sodium chloride (NaCl) crystal.

### Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply [Equation 9.30](#), [Equation 9.31](#), and [Equation 9.32](#) (along with the Pythagorean theorem).

### Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. [Figure 9.29](#) shows the coordinate system.



**Figure 9.29** A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so  $N = 8$ :

$$\vec{r}_{\text{CM}} = \frac{1}{M} \sum_{j=1}^8 m_j \vec{r}_j.$$

The mass of each of the chloride ions is

$$35.453\text{u} \times \frac{1.660 \times 10^{-27} \text{ kg}}{\text{u}} = 5.885 \times 10^{-26} \text{ kg}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26} \text{ kg}.$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26} \text{ kg}.$$

The total mass of the unit cell is therefore

$$M = (4)(5.885 \times 10^{-26} \text{ kg}) + (4)(3.816 \times 10^{-26} \text{ kg}) = 3.880 \times 10^{-25} \text{ kg}.$$

From the geometry, the locations are

$$\vec{r}_1 = 0$$

$$\vec{r}_2 = (2.36 \times 10^{-10} \text{ m}) \hat{i}$$

$$\vec{r}_3 = r_{3x} \hat{i} + r_{3y} \hat{j} = (2.36 \times 10^{-10} \text{ m}) \hat{i} + (2.36 \times 10^{-10} \text{ m}) \hat{j}$$

$$\vec{r}_4 = (2.36 \times 10^{-10} \text{ m}) \hat{j}$$

$$\vec{r}_5 = (2.36 \times 10^{-10} \text{ m}) \hat{k}$$

$$\vec{r}_6 = r_{6x} \hat{i} + r_{6z} \hat{k} = (2.36 \times 10^{-10} \text{ m}) \hat{i} + (2.36 \times 10^{-10} \text{ m}) \hat{k}$$

$$\vec{r}_7 = r_{7x} \hat{i} + r_{7y} \hat{j} + r_{7z} \hat{k} = (2.36 \times 10^{-10} \text{ m}) \hat{i} + (2.36 \times 10^{-10} \text{ m}) \hat{j} + (2.36 \times 10^{-10} \text{ m}) \hat{k}$$

$$\vec{r}_8 = r_{8y} \hat{j} + r_{8z} \hat{k} = (2.36 \times 10^{-10} \text{ m}) \hat{j} + (2.36 \times 10^{-10} \text{ m}) \hat{k}.$$

Substituting:

$$\begin{aligned} |\vec{r}_{\text{CM},x}| &= \sqrt{r_{\text{CM},x}^2 + r_{\text{CM},y}^2 + r_{\text{CM},z}^2} \\ &= \frac{1}{M} \sum_{j=1}^8 m_j (r_x)_j \\ &= \frac{1}{M} (m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\ &= \frac{1}{3.8804 \times 10^{-25} \text{ kg}} [(5.885 \times 10^{-26} \text{ kg})(0 \text{ m}) + (3.816 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) \\ &\quad + (5.885 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) \\ &\quad + (3.816 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) + 0 + 0 \\ &\quad + (3.816 \times 10^{-26} \text{ kg})(2.36 \times 10^{-10} \text{ m}) + 0] \\ &= 1.18 \times 10^{-10} \text{ m}. \end{aligned}$$

Similar calculations give  $r_{\text{CM},y} = r_{\text{CM},z} = 1.18 \times 10^{-10} \text{ m}$  (you could argue that this must be true, by symmetry, but it's a good idea to check).

### Significance

Although this is a great exercise to determine the center of mass given a Chloride ion at the origin, in fact the origin could be chosen at any location. Therefore, there is no meaningful application of the center of mass of a unit cell beyond as an exercise.



**9.12 Check Your Understanding** Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a *huge* number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in **Equation 9.29** to be zero. However, you must still include the mass of the object at your origin in your calculation of  $M$ , the total mass **Equation 9.19**. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.

## Center of Mass of Continuous Objects

If the object in question has its mass distributed uniformly in space, rather than as a collection of discrete particles, then  $m_j \rightarrow dm$ , and the summation becomes an integral:

$$\vec{r}_{\text{CM}} = \frac{1}{M} \int \vec{r} \, dm. \quad (9.34)$$

In this context,  $r$  is a characteristic dimension of the object (the radius of a sphere, the length of a long rod). To generate an integrand that can actually be calculated, you need to express the differential mass element  $dm$  as a function of the mass density of the continuous object, and the dimension  $r$ . An example will clarify this.

### Example 9.18

#### CM of a Uniform Thin Hoop

Find the center of mass of a uniform thin hoop (or ring) of mass  $M$  and radius  $r$ .

#### Strategy

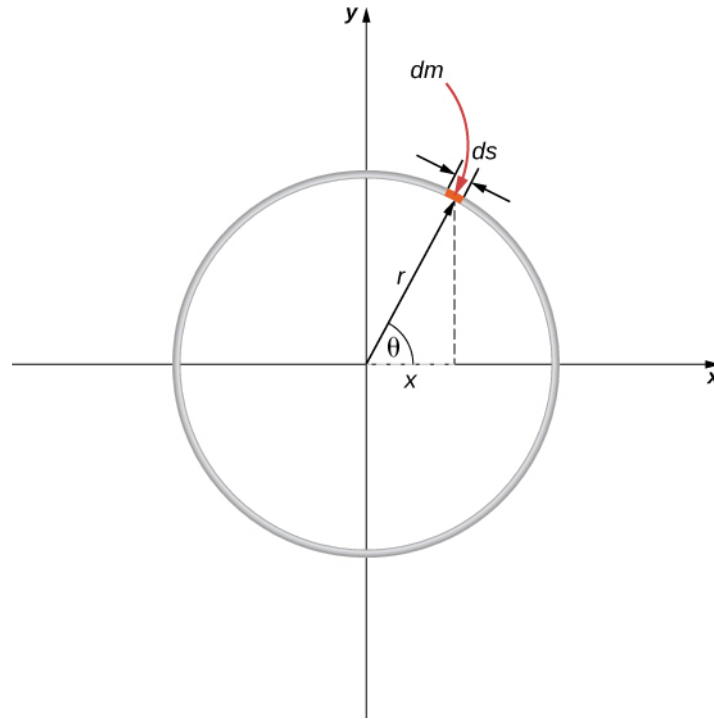
First, the hoop's symmetry suggests the center of mass should be at its geometric center. If we define our coordinate system such that the origin is located at the center of the hoop, the integral should evaluate to zero.

We replace  $dm$  with an expression involving the density of the hoop and the radius of the hoop. We then have an expression we can actually integrate. Since the hoop is described as "thin," we treat it as a one-dimensional object, neglecting the thickness of the hoop. Therefore, its density is expressed as the number of kilograms of material per meter. Such a density is called a **linear mass density**, and is given the symbol  $\lambda$ ; this is the Greek letter "lambda," which is the equivalent of the English letter "l" (for "linear").

Since the hoop is described as uniform, this means that the linear mass density  $\lambda$  is constant. Thus, to get our expression for the differential mass element  $dm$ , we multiply  $\lambda$  by a differential length of the hoop, substitute, and integrate (with appropriate limits for the definite integral).

#### Solution

First, define our coordinate system and the relevant variables (**Figure 9.30**).



**Figure 9.30** Finding the center of mass of a uniform hoop. We express the coordinates of a differential piece of the hoop, and then integrate around the hoop.

The center of mass is calculated with **Equation 9.34**:

$$\vec{r}_{\text{CM}} = \frac{1}{M} \int_a^b \vec{r} \, dm.$$

We have to determine the limits of integration  $a$  and  $b$ . Expressing  $\vec{r}$  in component form gives us

$$\vec{r}_{\text{CM}} = \frac{1}{M} \int_a^b \left[ (r \cos \theta) \hat{i} + (r \sin \theta) \hat{j} \right] dm.$$

In the diagram, we highlighted a piece of the hoop that is of differential length  $ds$ ; it therefore has a differential mass  $dm = \lambda ds$ . Substituting:

$$\vec{r}_{\text{CM}} = \frac{1}{M} \int_a^b \left[ (r \cos \theta) \hat{i} + (r \sin \theta) \hat{j} \right] \lambda ds.$$

However, the arc length  $ds$  subtends a differential angle  $d\theta$ , so we have

$$ds = r d\theta$$

and thus

$$\vec{r}_{\text{CM}} = \frac{1}{M} \int_a^b \left[ (r \cos \theta) \hat{i} + (r \sin \theta) \hat{j} \right] \lambda r d\theta.$$

One more step: Since  $\lambda$  is the linear mass density, it is computed by dividing the total mass by the length of the hoop:

$$\lambda = \frac{M}{2\pi r}$$

giving us

$$\begin{aligned}\vec{\mathbf{r}}_{\text{CM}} &= \frac{1}{M} \int_a^b \left[ (r \cos \theta) \hat{\mathbf{i}} + (r \sin \theta) \hat{\mathbf{j}} \right] \left( \frac{M}{2\pi r} \right) r d\theta \\ &= \frac{1}{2\pi} \int_a^b \left[ (r \cos \theta) \hat{\mathbf{i}} + (r \sin \theta) \hat{\mathbf{j}} \right] d\theta.\end{aligned}$$

Notice that the variable of integration is now the angle  $\theta$ . This tells us that the limits of integration (around the circular hoop) are  $\theta = 0$  to  $\theta = 2\pi$ , so  $a = 0$  and  $b = 2\pi$ . Also, for convenience, we separate the integral into the x- and y-components of  $\vec{\mathbf{r}}_{\text{CM}}$ . The final integral expression is

$$\begin{aligned}\vec{\mathbf{r}}_{\text{CM}} &= r_{\text{CM},x} \hat{\mathbf{i}} + r_{\text{CM},y} \hat{\mathbf{j}} \\ &= \left[ \frac{1}{2\pi} \int_0^{2\pi} (r \cos \theta) d\theta \right] \hat{\mathbf{i}} + \left[ \frac{1}{2\pi} \int_0^{2\pi} (r \sin \theta) d\theta \right] \hat{\mathbf{j}} \\ &= 0 \hat{\mathbf{i}} + 0 \hat{\mathbf{j}} = \vec{\mathbf{0}}\end{aligned}$$

as expected.

## Center of Mass and Conservation of Momentum

How does all this connect to conservation of momentum?

Suppose you have  $N$  objects with masses  $m_1, m_2, m_3, \dots, m_N$  and initial velocities  $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \dots, \vec{\mathbf{v}}_N$ . The center of mass of the objects is

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M} \sum_{j=1}^N m_j \vec{\mathbf{r}}_j.$$

Its velocity is

$$\vec{\mathbf{v}}_{\text{CM}} = \frac{d \vec{\mathbf{r}}_{\text{CM}}}{dt} = \frac{1}{M} \sum_{j=1}^N m_j \frac{d \vec{\mathbf{r}}_j}{dt} \quad (9.35)$$

and thus the initial momentum of the center of mass is

$$\begin{aligned}\left[ M \frac{d \vec{\mathbf{r}}_{\text{CM}}}{dt} \right]_i &= \sum_{j=1}^N m_j \frac{d \vec{\mathbf{r}}_{j,i}}{dt} \\ M \vec{\mathbf{v}}_{\text{CM},i} &= \sum_{j=1}^N m_j \vec{\mathbf{v}}_{j,i}.\end{aligned}$$

After these masses move and interact with each other, the momentum of the center of mass is

$$M \vec{\mathbf{v}}_{\text{CM},f} = \sum_{j=1}^N m_j \vec{\mathbf{v}}_{j,f}.$$

But conservation of momentum tells us that the right-hand side of both equations must be equal, which says

$$M \vec{\mathbf{v}}_{\text{CM},f} = M \vec{\mathbf{v}}_{\text{CM},i}. \quad (9.36)$$

This result implies that conservation of momentum is expressed in terms of the center of mass of the system. Notice

that as an object moves through space with no net external force acting on it, an individual particle of the object may accelerate in various directions, with various magnitudes, depending on the net internal force acting on that object at any time. (Remember, it is only the vector sum of all the internal forces that vanishes, not the internal force on a single particle.) Thus, such a particle's momentum will not be constant—but the momentum of the entire extended object will be, in accord with **Equation 9.36**.

**Equation 9.36** implies another important result: Since  $M$  represents the mass of the entire system of particles, it is necessarily constant. (If it isn't, we don't have a closed system, so we can't expect the system's momentum to be conserved.) As a result, **Equation 9.36** implies that, for a closed system,

$$\vec{v}_{\text{CM},f} = \vec{v}_{\text{CM},i} \quad (9.37)$$

That is to say, *in the absence of an external force, the velocity of the center of mass never changes.*

You might be tempted to shrug and say, “Well yes, that’s just Newton’s first law,” but remember that Newton’s first law discusses the constant velocity of a particle, whereas **Equation 9.37** applies to the center of mass of a (possibly vast) collection of interacting particles, and that there may not be any particle at the center of mass at all! So, this really is a remarkable result.

## Example 9.19

### Fireworks Display

When a fireworks rocket explodes, thousands of glowing fragments fly outward in all directions, and fall to Earth in an elegant and beautiful display (**Figure 9.31**). Describe what happens, in terms of conservation of momentum and center of mass.



**Figure 9.31** These exploding fireworks are a vivid example of conservation of momentum and the motion of the center of mass.

The picture shows radial symmetry about the central points of the explosions; this suggests the idea of center of mass. We can also see the parabolic motion of the glowing particles; this brings to mind projectile motion ideas.

### Solution

Initially, the fireworks rocket is launched and flies more or less straight upward; this is the cause of the more-or-less-straight, white trail going high into the sky below the explosion in the upper-right of the picture (the yellow explosion). This trail is not parabolic because the explosive shell, during its launch phase, is actually a rocket; the impulse applied to it by the ejection of the burning fuel applies a force on the shell during the rise-time interval. (This is a phenomenon we will study in the next section.) The shell has multiple forces on it; thus, it is not in free-fall prior to the explosion.

At the instant of the explosion, the thousands of glowing fragments fly outward in a radially symmetrical pattern.

The symmetry of the explosion is the result of all the internal forces summing to zero  $\left(\sum_j \vec{f}_j^{\text{int}} = 0\right)$ ; for every internal force, there is another that is equal in magnitude and opposite in direction.

However, as we learned above, these internal forces cannot change the momentum of the center of mass of the (now exploded) shell. Since the rocket force has now vanished, the center of mass of the shell is now a projectile (the only force on it is gravity), so its trajectory does become parabolic. The two red explosions on the left show the path of their centers of mass at a slightly longer time after explosion compared to the yellow explosion on the upper right.

In fact, if you look carefully at all three explosions, you can see that the glowing trails are not truly radially symmetric; rather, they are somewhat denser on one side than the other. Specifically, the yellow explosion and the lower middle explosion are slightly denser on their right sides, and the upper-left explosion is denser on its left side. This is because of the momentum of their centers of mass; the differing trail densities are due to the momentum each piece of the shell had at the moment of its explosion. The fragment for the explosion on the upper left of the picture had a momentum that pointed upward and to the left; the middle fragment's momentum pointed upward and slightly to the right; and the right-side explosion clearly upward and to the right (as evidenced by the white rocket exhaust trail visible below the yellow explosion).

Finally, each fragment is a projectile on its own, thus tracing out thousands of glowing parabolas.

### Significance

In the discussion above, we said, "...the center of mass of the shell is now a projectile (the only force on it is gravity)..." This is not quite accurate, for there may not be any mass at all at the center of mass; in which case, there could not be a force acting on it. This is actually just verbal shorthand for describing the fact that the gravitational forces on all the particles act so that the center of mass changes position exactly as if all the mass of the shell were always located at the position of the center of mass.



**9.13 Check Your Understanding** How would the firework display change in deep space, far away from any source of gravity?

You may sometimes hear someone describe an explosion by saying something like, "the fragments of the exploded object always move in a way that makes sure that the center of mass continues to move on its original trajectory." This makes it sound as if the process is somewhat magical: how can it be that, in *every* explosion, it *always* works out that the fragments move in just the right way so that the center of mass' motion is unchanged? Phrased this way, it would be hard to believe no explosion ever does anything differently.

The explanation of this apparently astonishing coincidence is: We defined the center of mass precisely so this is exactly what we would get. Recall that first we defined the momentum of the system:

$$\vec{p}_{\text{CM}} = \sum_{j=1}^N \frac{d\vec{p}_j}{dt}.$$

We then concluded that the net external force on the system (if any) changed this momentum:

$$\vec{F} = \frac{d\vec{p}_{\text{CM}}}{dt}$$

and then—and here's the point—we defined an acceleration that would obey Newton's second law. That is, we demanded

that we should be able to write

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}}{M}$$

which requires that

$$\vec{\mathbf{a}} = \frac{d^2}{dt^2} \left( \frac{1}{M} \sum_{j=1}^N m_j \vec{\mathbf{r}}_j \right)$$

where the quantity inside the parentheses is the center of mass of our system. So, it's not astonishing that the center of mass obeys Newton's second law; we defined it so that it would.

## 9.7 | Rocket Propulsion

### Learning Objectives

By the end of this section, you will be able to:

- Describe the application of conservation of momentum when the mass changes with time, as well as the velocity
- Calculate the speed of a rocket in empty space, at some time, given initial conditions
- Calculate the speed of a rocket in Earth's gravity field, at some time, given initial conditions

Now we deal with the case where the mass of an object is changing. We analyze the motion of a rocket, which changes its velocity (and hence its momentum) by ejecting burned fuel gases, thus causing it to accelerate in the opposite direction of the velocity of the ejected fuel (see **Figure 9.32**). Specifically: A fully fueled rocket ship in deep space has a total mass  $m_0$  (this mass includes the initial mass of the fuel). At some moment in time, the rocket has a velocity  $\vec{\mathbf{v}}$  and mass  $m$ ; this mass is a combination of the mass of the empty rocket and the mass of the remaining unburned fuel it contains. (We refer to  $m$  as the “instantaneous mass” and  $\vec{\mathbf{v}}$  as the “instantaneous velocity.”) The rocket accelerates by burning the fuel it carries and ejecting the burned exhaust gases. If the burn rate of the fuel is constant, and the velocity at which the exhaust is ejected is also constant, what is the change of velocity of the rocket as a result of burning all of its fuel?